

A COMBINATORIAL DESCRIPTION OF TOPOLOGICAL COMPLEXITY FOR FINITE SPACES

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ABSTRACT. This paper presents a combinatorial analog of topological complexity for finite spaces. We demonstrate that this coincides with the genuine topological complexity of the original finite space, and constitutes an upper bound for the topological complexity of its order complex.

1. INTRODUCTION

The *topological complexity* $\mathrm{TC}(X)$ of a space X is a homotopy invariant that measures how complex the space is. This invariant was introduced by Farber in the study of robotics motion planning [Far03]. Let us recall briefly the definition.

Definition 1.1. Let X be a path connected space, and let X^I be the path space of X consisting of continuous paths $\gamma : I = [0, 1] \rightarrow X$. The *path fibration* $p : X^I \rightarrow X \times X$ is defined by $p(\gamma) = (\gamma(0), \gamma(1))$. The *topological complexity* $\mathrm{TC}(X)$ is the smallest non-negative integer n such that there exists an open cover $\{U_i\}_{i=1}^n$ of $X \times X$ with a continuous section $s_i : U_i \rightarrow X^I$ of the path fibration p for each i . If such n does not exist, then we define $\mathrm{TC}(X) = \infty$.

This is a special case of a *sectional category* or *Schwarz genus* [Sch61] for the path fibration. In this paper, we focus on the topological complexity for finite T_0 spaces, which are regarded as finite partially ordered sets (posets for short). We introduce another invariant $\mathrm{CC}(P)$, called the *combinatorial complexity* for a finite space P using purely combinatorial terms. Our main aim is to show the following equality.

Theorem A (Theorem 3.6). It holds that $\mathrm{TC}(P) = \mathrm{CC}(P)$ for any path connected finite space P .

This theorem suggests us that the topological complexity of a finite space can be calculated using combinatorial methods.

On the other hand, a finite space P can be associated with a simplicial complex $\mathcal{K}(P)$, called the *order complex*. The combinatorial complexity of a finite space P is an upper bound for the topological complexity of the order complex $\mathcal{K}(P)$.

Theorem B (Theorem 4.4). It holds that $\mathrm{TC}(\mathcal{K}(P)) \leq \mathrm{CC}(P)$ for a path connected finite space P .

The remainder of this paper is organized as follows. Section 2 provides the definition of combinatorial complexity for finite spaces, including some basic homotopical properties of finite spaces. In Section 3, we prove the equality between the combinatorial and topological complexity of a finite space. Section 4 compares the combinatorial complexity of a finite space and the topological complexity of its order complex.

2. COMBINATORIAL COMPLEXITY FOR FINITE SPACES

This paper focuses on finite topological spaces consisting of finite points. We often consider a finite topological space as a discrete space, because every finite T_1 space must be discrete. In contrast, finite T_0 spaces play an important role in homotopy theory for finite complexes. In particular, every finite simplicial complex has the weak homotopy type of a finite T_0 space. On the other hand, T_0 -Alexandorff spaces are closely related to posets. A T_0 -Alexandorff space is equipped with a partial order $x \leq y$ defined by $x \in U_y$, where U_y is the smallest open set containing y . Conversely, a poset is equipped with the Alexandorff topology generated from its ideals. Here, an ideal of a poset P is a subset Q satisfying $x \in Q$ whenever $x \leq y$ for some $y \in Q$. From the above viewpoint, we can identify a T_0 -Alexandorff space with a poset. In particular, a finite T_0 space can be regarded as a finite poset. Let us simply call this a *finite space*. A finite space can be associated with a simplicial complex, called the *order complex*.

Definition 2.1. Let P be a finite space. The *order complex* $\mathcal{K}(P)$ is a simplicial complex whose n -simplexes are ordered sequences $p_0 < \dots < p_n$ in P .

Now, we propose a combinatorial analog of topological complexity for finite spaces. Let J_m denote the finite space consisting of $m + 1$ points with the zigzag order,

$$0 < 1 > 2 < \dots > (<)m.$$

This finite space is called the *finite fence* with length m , and behaves as an interval in terms of finite spaces. We refer the reader to [Sto66] and [Bar11] for the homotopy theory of finite spaces. A finite space P is path connected if and only if for any $x, y \in P$, there exists $m \geq 0$ and a continuous map $\gamma : J_m \rightarrow P$ such that $\gamma(0) = x$ and $\gamma(m) = y$. More generally, two maps $f, g : P \rightarrow Q$ between finite spaces are homotopic if and only if there exists $m \geq 0$ and a continuous map $H : P \times J_m \rightarrow Q$ such that $H_0 = f$ and $H_m = g$. By the exponential law, this is equivalent to considering a continuous map $H' : J_m \rightarrow Q^P$ such that $H'(0) = f$ and $H'(m) = g$.

Definition 2.2. Let P be a finite space. A *combinatorial path* of P with length $m \geq 0$ is a continuous map $\gamma : J_m \rightarrow P$. Note that a map between finite spaces is continuous if and only if it is order-preserving (a poset map). From this reason, a combinatorial path is a zigzag sequence of P formed as follows:

$$x_0 \leq x_1 \geq x_2 \leq \dots \geq (\leq)x_m.$$

Let P^{J_m} denote the finite space of combinatorial paths of P with length m , equipped with the pointwise order.

Note that the Alexandorff topology on P^{J_m} coincides with the compact open topology, by Kukiela's result (Corollary 4.7 of [Kuk10]). As an analog of path fibration, it is equipped with the canonical continuous map $q_m : P^{J_m} \rightarrow P \times P$ given by $q_m(\gamma) = (\gamma(0), \gamma(m))$ for each $m \geq 0$.

Definition 2.3. Let P be a path connected finite space. For $m \geq 0$, define $\text{CC}_m(P)$ as the smallest non-negative integer n such that there exists an open cover $\{Q_i\}_{i=1}^n$ of $P \times P$ with a continuous section $s_i : Q_i \rightarrow P^{J_m}$ of q_m for each i . If such n does not exist, then we define $\text{CC}_m(P) = \infty$.

Lemma 2.4. For any $m \geq 0$ and path connected finite space P , it holds that $\text{CC}_{m+1}(P) \leq \text{CC}_m(P)$.

Proof. Assume that $CC_m(P) = n$. Then, there exists an open cover $\{Q_i\}_{i=1}^n$ of $P \times P$ with a continuous section $s_i : Q_i \rightarrow P^{J_m}$ of q_m for each i . The retraction $r : J_{m+1} \rightarrow J_m$ sending $m+1$ to m induces a map $r^* : P^{J_m} \rightarrow P^{J_{m+1}}$ such that the following diagram commutes:

$$\begin{array}{ccc} P^{J_m} & \xrightarrow{r^*} & P^{J_{m+1}} \\ & \searrow q_m \quad \swarrow q_{m+1} & \\ & P \times P & \end{array}$$

The composition $r^* \circ s_i : Q_i \rightarrow P^{J_{m+1}}$ is a continuous section of q_{m+1} for each i . Thus, $CC_{m+1}(P) \leq n$. \square

The topological complexity is closely related to the LS-category of a space. The LS-category $\text{cat}(X)$ of a space X is the smallest non-negative integer n such that X can be covered by n open subspaces that are contractible in X . Faber proved that the following inequality holds for suitable spaces X (Theorem 5 in [Far03]):

$$\text{cat}(X) \leq \text{TC}(X) \leq \text{cat}(X \times X) \leq 2\text{cat}(X) - 1.$$

Let us consider the case of finite spaces.

Lemma 2.5. *It holds that $CC_m(P) \leq \text{cat}(P \times P)$ for a sufficiently large $m \geq 0$ and a path connected finite space P .*

Proof. Assume that $\text{cat}(P \times P) = n$. Then, there exists a contractible open cover $\{Q_i\}_{i=1}^n$ in $P \times P$. For each i , fix an element $(x_1, x_2) \in Q_i$ and a path $\gamma : J_k \rightarrow P$ between x_1 and x_2 , and choose a contracting homotopy $H : Q_i \times J_\ell \rightarrow P \times P$ onto (x_1, x_2) . The first and second projections yield two maps, $H_1, H_2 : Q_i \rightarrow P^{J_\ell}$, such that $H_j(a_1, a_2)(0) = a_j$ and $H_j(a_1, a_2)(\ell) = x_j$ for $j = 1, 2$. Let m_i denote $2\ell + k$. A continuous section $s_i : Q_i \rightarrow P^{J_{m_i}}$ of p_{m_i} is defined by the concatenation of paths $s_i(a_1, a_2) = H_1(a_1, a_2) * \gamma * H_2(a_1, a_2)^{-1}$. Hence, we obtain a continuous section $Q_i \rightarrow P^{J_m}$ of p_m for $m = \max\{m_i\}_{i=1}^n$. Thus, $CC_m(P) \leq n$. \square

For a finite space P , the *prime ideal* at $x \in P$ is the ideal formed of $\{y \in P \mid y \leq x\}$. This is contractible onto x . If $\text{Max}(P) = \{x_1, \dots, x_k\}$ is the set of maximal points of P , then their prime ideals become a contractible open cover of P . This implies the inequality $\text{cat}(P) \leq k = \text{Max}(P)^\sharp$. The next corollary follows from the fact that $\text{Max}(P \times P) = \text{Max}(P) \times \text{Max}(P)$.

Corollary 2.6. *It holds that $CC_m(P) \leq (\text{Max}(P)^\sharp)^2$ for a sufficiently large $m \geq 0$ and a path connected finite space P .*

Remark 2.7. Note that the product formula

$$\text{cat}(P \times Q) \leq \text{cat}(P) + \text{cat}(Q) - 1$$

does **not** hold in general for finite spaces P, Q . This requires the spaces to be paracompact Hausdorff spaces (occasionally a paracompact space is defined such that it is always Hausdorff) with a partition of unity for every finite open cover (see Proposition 2.3 of [Jam78]). For this reason, the equality $\text{cat}(P \times P) \leq 2\text{cat}(P) - 1$ does not hold for an arbitrary finite space P .

Lemma 2.4 and Corollary 2.6 allow us to define the notion of combinatorial complexity for finite spaces.

Definition 2.8. Let P be a path connected finite space. The *combinatorial complexity* $CC(P)$ is the minimum of $CC_m(P)$:

$$CC(P) = \min\{CC_m(P)\}_{m=0}^\infty = \lim_{m \rightarrow \infty} CC_m(P).$$

3. TOPOLOGICAL AND COMBINATORIAL COMPLEXITY OF A FINITE SPACE

Let us examine the relationship between topological and combinatorial complexity. Using the canonical map $\mathcal{K}(P) \rightarrow P$ given by McCord [McC66], we can obtain the next inequality.

Proposition 3.1. *It holds that $TC(P) \leq CC(P)$ for a path connected finite space P .*

Proof. Assume that $CC(P) = n$. Then, there exists an open cover $\{Q_i\}_{i=1}^n$ of $P \times P$ with a continuous local section $s_i : Q_i \rightarrow P^{J_m}$ of q_m for each i and some $m \geq 0$. Let $\alpha_m : [0, m] \cong \mathcal{K}(J_m) \rightarrow J_m$ denote the canonical map given by McCord [McC66]. This map is defined by

$$\alpha_m(t) = \begin{cases} 2k-1 & (t = 2k-1), \\ 2k & (2k-1 < t < 2k+1), \end{cases}$$

for $k = 0, 1, \dots$. In particular, this map preserves both ends, i.e., $\alpha_m(0) = 0$ and $\alpha_m(m) = m$. Let $\beta : I \rightarrow J_m$ denote the composition of α_m and the m -times isomorphism $I = [0, 1] \cong [0, m]$. This induces $\beta^* : P^{J_m} \rightarrow P^I$, such that the following diagram is commutative:

$$\begin{array}{ccc} P^{J_m} & \xrightarrow{\beta^*} & P^I \\ & \searrow q_m & \swarrow p \\ & P \times P & \end{array}$$

The composition $\alpha^* \circ s_i : Q_i \rightarrow P^I$ is a continuous section of the path fibration p for each i . Thus, $TC(P) \leq n$. \square

Before discussing the converse inequality of the above, we recall Iwase and Sakai's work on LS-category and topological complexity [IS10]. They described the topological complexity of a space B as a *fiberwise unpointed LS-category* of the fiberwise pointed space $B \times B$ over B .

Definition 3.2. Let X be a fiberwise pointed space over B with a projection $p : X \rightarrow B$ and a section $s : B \rightarrow X$. The *fiberwise unpointed LS-category* $\text{cat}_B^*(X)$ is the smallest non-negative integer n such that there exists an open cover $\{U_i\}_{i=1}^n$ of $B \times B$ each of which is fiberwise compressible into $s(B)$ in X by a fiberwise homotopy. If such n does not exist, define $\text{cat}_B^*(X) = \infty$.

Note that we use the unreduced version of fiberwise unpointed LS-category, for simplicity, i.e., our definition is equal to the original one plus 1.

Theorem 3.3 (Theorem 1.7 [IS10]). *Let B be a path connected space. Consider the fiberwise pointed space $B \times B$ over B with the second projection $\text{pr}_2 : B \times B \rightarrow B$ and the diagonal $\Delta : B \rightarrow B \times B$. Then, we have $TC(B) = \text{cat}_B^*(B \times B)$.*

Lemma 3.4. *Let R be a finite space, and P, Q be fiberwise finite spaces over R . If two maps $f, g : P \rightarrow Q$ are fiberwise homotopic over R , then we can take a map $H : P \times J^m \rightarrow Q$ over R such that $H_0 = f$ and $H_m = g$ for some $m \geq 0$.*

Proof. Let Q_R^P denote the space of maps from P to Q over R as a subspace of Q^P . A fiberwise homotopy between f and g gives a continuous path $I \rightarrow Q_R^P$. Since the space Q_R^P is also a finite space, we can take a combinatorial path $J_m \rightarrow Q_R^P$ starting at f and ending at g for some $m \geq 0$. It determines our desired continuous map $H : P \times J_m \rightarrow Q$ over R such that $H_0 = f$ and $H_1 = g$. \square

Proposition 3.5. *It holds that $\text{CC}(P) \leq \text{cat}_P^*(P \times P)$ for any path connected finite space P .*

Proof. Assume that $\text{cat}_P^*(P \times P) = n$. There exists an open cover $\{Q_i\}_{i=1}^n$ of $P \times P$ each of which is fiberwise compressible into $\Delta(P)$ by a fiberwise homotopy $Q_i \times I \rightarrow P \times P$. By Lemma 3.4, we have a continuous map $H : Q_i \times J_m \rightarrow P \times P$ over P such that $H(x, y, 0) = (x, y)$ and $H(x, y, m) = (y, y)$ for some $m \geq 0$. Composing with the first projection $\text{pr}_1 : P \times P \rightarrow P$, we obtain $H_1 : Q_i \times J_m \rightarrow P$. It yields a continuous section $Q_i \rightarrow P^{J_m}$ of q_m . Thus, $\text{CC}(P) \leq n$. \square

Proposition 3.5 and 3.1, and Theorem 3.3 conclude our main result.

Theorem 3.6. *It holds that $\text{TC}(P) = \text{CC}(P)$ for any path connected finite space P .*

Corollary 3.7. *The following basic homotopical properties hold for combinatorial complexity. These have been shown in [Far03] as properties for topological complexity.*

- A finite space P is contractible if and only if $\text{CC}(P) = 1$.
- Combinatorial complexity depends only on the homotopy type of finite spaces, i.e., $\text{CC}(P) = \text{CC}(Q)$ if two finite spaces P and Q are homotopy equivalent to each other (in other words, they have isomorphic cores [Sto66]).

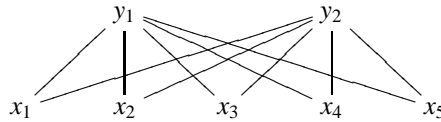
The next corollary follows from combining Theorem 3.6 with Corollary 2.6 and Farber's result.

Corollary 3.8. *The following inequalities hold for a path connected finite space P :*

$$\text{cat}(P) \leq \text{TC}(P) = \text{CC}(P) \leq \text{cat}(P \times P) \leq (\text{Max}(P)^\#)^2.$$

For a finite space P , the *opposite space* P^{op} is the finite space consisting of the same underlying set as P with the reversed partial order of P .

Remark 3.9. In general, $\text{CC}(P) \neq \text{CC}(P^{\text{op}})$. Consider the finite space P described by the following Hasse diagram:



This is a slightly modified version of Example 5.2 in [FMV15]. We have that $\text{cat}(P) = \text{Max}(P)^\# = 2$, but $\text{cat}(P^{\text{op}}) = \text{Min}(P)^\# = 5$. Corollary 3.8 implies that

$$\text{CC}(P) \leq \text{cat}(P \times P) \leq 4 < 5 = \text{cat}(P^{\text{op}}) \leq \text{CC}(P^{\text{op}}).$$

Thus, $\text{CC}(P) \neq \text{CC}(P^{\text{op}})$.

4. TOPOLOGICAL AND COMBINATORIAL COMPLEXITY OF THE ORDER COMPLEX

Next, we focus on the relationship between the combinatorial complexity of a finite space P and the topological complexity of the order complex $\mathcal{K}(P)$. We first introduce a slightly modified definition of topological complexity for CW complexes, using their subcomplexes.

Definition 4.1. Let X be a path connected CW complex. The *topological cellular complexity* $\text{TCC}(X)$ is the smallest non-negative integer n such that there exists a cover $\{Y_i\}_{i=1}^n$ consisting of subcomplexes of $X \times X$, with a continuous local section $s_i : Y_i \rightarrow X^I$ of the path fibration p for each i . If such n does not exist, then we define $\text{TCC}(X) = \infty$.

Proposition 4.2. *It holds that $\text{TC}(X) \leq \text{TCC}(X)$ for a path connected CW complex X .*

Proof. If $\text{TCC}(X) = n$, then there exist subcomplexes $\{Y_i\}_{i=1}^n$ covering $X \times X$, and a continuous local section $s_i : Y_i \rightarrow X^I$ of the path fibration p for each i . Note that a subcomplex of X is a neighborhood of deformation retract of X . We can take an open neighborhood U_i of Y_i with a deformation retraction $r_i : U_i \rightarrow Y_i$, and a homotopy $H_i : U_i \times I \rightarrow U_i$ between r_i and the identity map on U_i . For $x = (x_1, x_2) \in U_i \subset X \times X$, the path $\gamma = H_i(x, -) : I \rightarrow U_i$ starts at x and ends at $r_i(x) \in Y_i$. Let γ_j denote the path on X defined by composing the projection pr_j and γ , for $j = 1, 2$. The concatenation $\gamma_1 * s_i(r_i(x)) * \gamma_2^{-1}$ is a path starting at x_1 and ending at x_2 , where $\gamma_2^{-1}(t) = \gamma_2(1 - t)$. This construction extends s_i on Y_i to a continuous section on U_i for each i . Thus, $\text{TC}(X) \leq n$. \square

Theorem 4.3. *It holds that $\text{TCC}(\mathcal{K}(P)) \leq \text{CC}(P)$ for a path connected finite space P .*

Proof. Assume that $\text{CC}(P) = n$. There exists an open cover $\{Q_i\}_{i=1}^n$ of $P \times P$, with a local section $s_i : Q_i \rightarrow P^{J_m}$ of q_m for some $m \geq 0$. Note that we have the canonical isomorphism $\pi : \mathcal{K}(P \times P) \cong \mathcal{K}(P) \times \mathcal{K}(P)$ induced from the projection $P \times P \rightarrow P$. The images $\pi(\mathcal{K}(Q_1)), \dots, \pi(\mathcal{K}(Q_n))$ are subcomplexes covering $\mathcal{K}(P) \times \mathcal{K}(P)$. Here, we regard $\mathcal{K}(P) \times \mathcal{K}(P)$ as a cell complex, whose cells are the products of simplices in $\mathcal{K}(P)$. Consider the map $\varphi : \mathcal{K}(P^{J_m}) \rightarrow \mathcal{K}(P)^I$ induced from the following composition map:

$$\mathcal{K}(P^{J_m}) \times I \cong \mathcal{K}(P^{J_m}) \times [0, m] \cong \mathcal{K}(P^{J_m}) \times \mathcal{K}(J_m) \cong \mathcal{K}(P^{J_m} \times J_m) \xrightarrow{\mathcal{K}(\text{ev})} \mathcal{K}(P).$$

Here, the final map is induced from the evaluation map $\text{ev} : P^{J_m} \times J_m \rightarrow P$. This map φ makes the following right diagram commute:

$$\begin{array}{ccccc} \mathcal{K}(Q_i) & \xrightarrow{\mathcal{K}(s_i)} & \mathcal{K}(P^{J_m}) & \xrightarrow{\varphi} & \mathcal{K}(P)^I \\ \parallel & & \downarrow \mathcal{K}(q_m) & & \downarrow p \\ \mathcal{K}(Q_i) & \hookrightarrow & \mathcal{K}(P \times P) & \xrightarrow{\pi} & \mathcal{K}(P) \times \mathcal{K}(P). \end{array}$$

We obtain a continuous section $\pi(\mathcal{K}(Q_i)) \rightarrow \mathcal{K}(P)^I$ of the path fibration p as the following composition:

$$\pi(\mathcal{K}(Q_i)) \xrightarrow{\pi^{-1}} \mathcal{K}(Q_i) \xrightarrow{\mathcal{K}(s_i)} \mathcal{K}(P^{J_m}) \xrightarrow{\varphi} \mathcal{K}(P)^I.$$

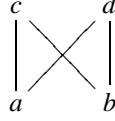
Thus, $\text{TCC}(\mathcal{K}(P)) \leq n$. \square

The above theorem and Proposition 4.2 imply the next inequality.

Theorem 4.4. *It holds that $\text{TC}(\mathcal{K}(P)) \leq \text{CC}(P)$ for a path connected finite space P .*

Finally, let us compute the topological and combinatorial complexity of the minimal model P of a circle S^1 . Farber's result (Theorem 8 [Far03]) implies that $\text{TC}(\mathcal{K}(P)) = \text{TC}(S^1) = 2$. By Theorem 4.4, we have that $\text{TC}(P) = \text{CC}(P) \geq \text{TC}(\mathcal{K}(P)) = 2$. The next example demonstrates a case that the above inequality is strict.

Example 4.5. Let P be a finite space consisting of four points a, b, c, d described as the following Hasse diagram:



We will show that $\text{TC}(P) = \text{CC}(P) = 4$. The finite space P is not contractible with two maximal points, hence Corollary 3.8 implies the inequality $2 \leq \text{TC}(P) = \text{CC}(P) \leq 4$. Assume that $\text{CC}(P) \leq 3$. There exists an open set (ideal) Q of $P \times P$ including at least two distinguished maximal points (p_1, p_2) and (p'_1, p'_2) of $P \times P$, with a continuous section $s : Q \rightarrow P^{J_m}$ of q_m for some $m \geq 0$. Note that Q contains all minimal points of $P \times P$. Consider the path $s(\alpha, \beta) : J_m \rightarrow P$ for $(\alpha, \beta) \in \{(a, b), (b, a)\}$. This must pass through a maximal point c or d to combine the distinguished minimal points α and β . Let $u_{(\alpha, \beta)}$ and $v_{(\alpha, \beta)}$ be the minimal and maximal values, respectively, of the inverse image $s(\alpha, \beta)^{-1}(\{c, d\})$. If we write $u = \min\{u_{(a, b)}, u_{(b, a)}\}$ and $v = \max\{v_{(a, b)}, v_{(b, a)}\}$, then $s(\alpha, \beta)(t) = \alpha$ and $s(\alpha, \beta)(s) = \beta$ for any $0 \leq t < u$ and $v < s \leq m$. The continuity of the section states that $s(\alpha, \beta) \leq s(\gamma, \delta)$ for any $(\gamma, \delta) \in \{(p_1, p_2), (p'_1, p'_2)\}$. Thus, it turns out that $s(\gamma, \delta)(t)$ is never a minimal point of P for any $0 \leq t < u, v < t \leq m$. Furthermore, $s(p_1, p_2)(w) = s(p'_1, p'_2)(w)$ for each $w \in \{u, v\}$. This implies that $s(p_1, p_2)(t) = s(p'_1, p'_2)(t)$ for any $0 \leq t \leq u$ and $v \leq t \leq m$. The equality $(p_1, p_2) = (p'_1, p'_2)$ holds. However, this contradicts our assumption. Thus, $\text{CC}(P) \not\leq 3$, and so $\text{CC}(P) = \text{TC}(P) = 4$.

Concluding remarks and future work. This paper introduced a combinatorial analog of topological complexity, and studied the relationship between the two. This is a first step towards exploring combinatorial methods of calculating the topological complexity of finite spaces and their associated order complexes. We will mention some possible future directions suggested by this paper.

- Faber employed the cup-product of the cohomology of a space to compute the topological complexity. The *zero-divisors-cup-length* (Definition 6 in [Far03]) provides a useful lower bound for topological complexity. When P is a finite space, the zero-divisors-cup-length z_P of P is equal to $z_{\mathcal{K}(P)}$ of the order complex $\mathcal{K}(P)$, because P and $\mathcal{K}(P)$ are weakly homotopy equivalent and have isomorphic cohomologies. This implies the following inequality:

$$z_P = z_{\mathcal{K}(P)} \leq \text{TC}(\mathcal{K}(P)) \leq \text{TC}(P) = \text{CC}(P).$$

This suggests that z_P is not a better lower bound than $\text{TC}(\mathcal{K}(P))$ for $\text{CC}(P)$ and $\text{TC}(P)$. Indeed, this was not effective in the calculation in Example 4.5. For calculating the combinatorial complexity, we must develop other techniques that are suited to finite spaces. This will help to compute more complex cases than that in Example 4.5.

- There are several variant types of topological complexity. For example, symmetric version [FG07], higher version [Rud10], [BGRT14], and monoidal version [IS10]. Combinatorial analogs of them may be interesting to explore.
- Faber originally introduced the notion of topological complexity for application to the problem of motion planning for a robot arm. We hope that our combinatorial complexity can also be applied to such practical problems.

REFERENCES

- [Are99] F. G. Arenas. Alexandroff spaces. *Acta Math. Univ. Comenian. (N.S.)* 68 (1999), no. 1, 17–25.

- [Bar11] J. A. Barmak. *Algebraic topology of finite topological spaces and applications*. Lecture Notes in Mathematics, 2032. Springer, Heidelberg, 2011. xviii+170 pp.
- [BGRT14] I. Basabe, J. González, Y. B. Rudyak, D. Tamaki. Higher topological complexity and its symmetrization. *Algebr. Geom. Topol.* 14 (2014), no. 4, 2103–2124
- [Far03] M. Farber. Topological complexity of motion planning *Discrete Comput. Geom.* 29 (2003), no. 2, 211–221.
- [FG07] M. Farber, M. Grant. Symmetric motion planning. *Topology and robotics*, 85–104, Contemp. Math., 438, Amer. Math. Soc., Providence, RI, 2007.
- [FMV15] D. Fernández-Ternero and E. Macías-Virgós and J. A. Vilches. Lusternik–Schnirelmann category of simplicial complexes and finite spaces. *Topology Appl.* 194 (2015), 37–50.
- [IS10] N. Iwase, M. Sakai. Topological complexity is a fibrewise L-S category. *Topology Appl.* 157 (2010), no. 1, 10–21.
- [Jam78] I. M. James. On category, in the sense of Lusternik–Schnirelmann. *Topology* 17 (1978), no. 4, 331–348.
- [Kuk10] M. Kukiela. On homotopy types of Alexandroff spaces. *Order* 27 (2010), no. 1, 9–21.
- [McC66] M. C. McCord. Singular homology groups and homotopy groups of finite topological spaces. *Duke Math. J.* 33 (1966) 465–474.
- [Rud10] Y. B. Rudyak. On higher analogs of topological complexity. *Topology Appl.* 157 (2010), no. 5, 916–920.
- [Sch61] A. S. Schwarz. The genus of a fiber space. *Tr. Mosk. Mat. Obs.* 10 (1961), 217–272.
- [Sto66] R. E. Stong. Finite topological spaces. *Trans. Amer. Math. Soc.* 123 (1966) 325–340.

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